A journey of a thousand miles begins with a single step.

Lao Tsu
Hoare Triples

\[ C = \text{def} \quad C.\text{Target} := C.\text{Source} : \frac{\text{true}}{\{Q[c.\text{Source} \ \backslash \ C.\text{Target}]\}}\ C \ \{Q\} \]

\[ C = \text{def} \quad \text{if } C.\text{Test} \text{ then } C.\text{Then} : \frac{\{P \land C.\text{Test}\}C.\text{Then}\{Q\}}{\{P\}C\{Q\}} \]

\[ C = \text{def} \quad \text{if } C.\text{Test} \text{ then } C.\text{Then} \text{ else } C.\text{Else} : \frac{\{P \land C.\text{Test}\}C.\text{Then}\{Q\}}{\{P\}C\{Q\}} \]

\[ C = \text{def} \quad C_1; C_2 : \frac{\{P\}C_1\{R\} \quad \{R\}C_2\{Q\}}{\{P\}C_1; C_2\{Q\}} \]

\{a > 4 \}

C_1: x := a + 2;

C_2: y := x - 2;

C_3: z := y + 2;

\{a > 4, x > 6, y > 4, z > 6\}

\[ \begin{array}{c}
\text{true} \\
{a > 4}\ C_1\{S\} \\
\{S\} C_2\{R\} \\
{a > 4}\ C_1; C_2\{R\} \\
{a > 4, x > 6, y > 4, z > 6}\ C_3\{a > 4, x > 6, y > 4, z > 6\} \\
{a > 4}\ (C_1; C_2); C_3\{a > 4, x > 6, y > 4, z > 6\} \\
\{R\} z := y + 2\{a > 4, x > 6, y > 4, z > 6\} \\
\{R\} = \{a > 4, x > 6, y > 4, z > 6\} [y + 2 \backslash z] \\
\quad = \{a > 4, x > 6, y > 4, y + 2 > 6\} \\
\quad = \{a > 4, x > 6, y > 4, y > 6 - 2\} = \{a > 4, x > 6, y > 4\} \\
\{S\} y := x - 2\{a > 4, x > 6, y > 4\} \\
\{S\} = \{a > 4, x > 6, y > 4\} [x - 2 \backslash y] \\
\quad = \{a > 4, x > 6, x - 2 > 4\} = \{a > 4, x > 6\} \\
{a > 4} x := a + 2\{a > 4, x > 6\} \\
{a > 4} = \{a > 4, x > 6\} [a + 2 \backslash x] = \{a > 4, a + 2 > 6\} = \{a > 4\} \]
Problem 1

Validate

\{a =4, b>5\}

x := a + b * 3;

y := x - 10;

\{a=4, b>5, x>19, y>9\}

Solution:

P = \{a =4, b>5\}

C1 = x := a + b * 3;

C2 = y := x - 10;

Q = \{a=4, b>5, x>19, y>9\}

\{Q[x-10 \ y]\} y := x - 10\{Q\}

\{a=4, b>5, x>19, y>9\}[x-10 \ y]=\{a=4, b>5, x>19, x-10>9\}=\{a=4, b>5, x>19\}

P=\{a=4, b>5, x>19\}[a+b*3\ x] x := a + b * 3; \{a=4, b>5, x>19\}

P=\{a=4, b>5, a+b*3>19\}=\{a=4, b=5, b>(19-a)/3\}=\{a=4, b>5, b>(19-4)/3\}=\{a=4, b>5\}
Problem 2

Validate

\{x=b\}

if \ a >= \ b then \ x := \ a;

\{x=max(a,b)\}

Solution:

\[ a >= b \rightarrow a = \max(a,b) \quad \frac{\text{true}}{\{x = b, a >= b, a = \max(a,b)\} \ x := a; \ {x = \max(a,b)} \} \]

\[ a < b \rightarrow b = \max(a,b) \quad \frac{\{x = b, a >= b, b = \max(a,b)\} \ \rightarrow \ \{x = \max(a,b)\}}{\{x = b\} \text{if} \ a >= b \ \text{then} \ x := a; \ \{x = \max(a,b)\}} \]
Problem 3

Validate

\{true\}

if x < 0 then val := -x; else val := x;
\{val=abs(x)\}

Solution:

\begin{align*}
\text{if } x < 0 \rightarrow x &= -\text{abs}(x) \\
\text{true} &\quad \{x < 0, x = -\text{abs}(x)\} \text{ val := } -x; \quad \{\text{val} = \text{abs}(x)\}
\end{align*}

\begin{align*}
\text{if } x \geq 0 \rightarrow x &= \text{abs}(x) \\
\text{true} &\quad \{x \geq 0, x = \text{abs}(x)\} \text{ val := } x; \quad \{\text{val} = \text{abs}(x)\}
\end{align*}

\{true\} if x < 0 then val := -x; else val := x; \{val = \text{abs}(x)\}
Induction

- There are at most \( m^h \) leaves in an \( m \)-ary tree of height \( h \).

  **Proof:** We use mathematical induction on the height of trees.

  **Base Step:**
  Consider \( m \)-ary trees of height 1. These trees consist of a root with no more than \( m \) children, each of which is a leaf. Hence there are no more than \( m^1 = m \) leaves in an \( m \)-ary tree of height 1.

  **Induction Step:**
  Assume that the result is true for all \( m \)-ary trees of height \( k \) less than \( h \). Let \( t \) be an \( m \)-ary tree of height \( h \). The leaves of \( t \) are the leaves of the subtrees of \( t \) obtained by deleting the edges from the root to each of the vertices at level 1. Now, each subtree has a height less than or equal to \( h-1 \). So by the induction hypothesis, each of these rooted trees has at most \( m^{h-1} \) leaves. Since there are at most \( m \) such subtrees, each with a maximum of \( m^{h-1} \) leaves, there are at most \( m \times m^{h-1} = m^h \) leaves in the rooted tree.

  Q.E.D.

- Consider the following grammar:

  \[ e ::= 0 \mid 2 \mid e + e \mid e * e \]

  Show that the value of every expression produced by this grammar is an even number.

  **Proof:** We use induction on the structure of expressions.

  \( e = 0 \): immediate

  \( e = 2 \): immediate

  \( e = e_1 + e_2 \): By the induction hypothesis, both \( e_1 \) and \( e_2 \) are elements of \( e \). Hence they produce even numbers. Let \( n_1 = 2k \) and \( n_2 = 2m \) be those even numbers. Then we have \( 2k + 2m = 2(k + m) \), which is an even number as required.

  \( e = e_1 * e_2 \): By the induction hypothesis, both \( e_1 \) and \( e_2 \) are elements of \( e \). Hence they produce even numbers. Let \( n_1 = 2k \) and \( n_2 = 2m \) be those even numbers. Then we have \( 2k \times 2m = 2(k * m) \), which is an even number as required.

  Q.E.D.
**Problem 1**

Use mathematical induction to prove that \( n! < n^n \) whenever \( n > 1 \).

**Solution:**

**Base case** \( n = 2 \):

\[
2! < 2^2 \\
1 \times 2 < 2 \times 2 \\
2 < 4
\]

**Inductive Step:**
- Assume \( n! < n^n \) if \( n > 2 \)
- We need to show \((n+1)! < (n+1)^{(n+1)}\) if \( n > 2 \)

\[
(n+1)! = (n+1)n! < (n+1)n^n \\leq (n+1)(n+1)^n = (n+1)^{(n+1)}
\]

Q.E.D.
Problem 2

Consider the following grammar:

\[ t ::= \text{nil} \mid \text{leaf} \mid \text{node}(t, t) \]

Use structural induction on trees (with two base cases and one induction step) to prove that the size of a binary tree is at most \(2^h\), where \(h\) is the height of the tree.

Solution:

Base case \(t = \text{nil}\): vacuously true, there are no trees with \(h = 0\).

Base case \(t = \text{leaf}\): we have \(h = 1\), size = 1 < 2^1.

Inductive step \(t = \text{node}(t_1, t_2)\):

Assume the result is true for all trees of height \(k < h\). Let \(t\) be a tree of height \(h\). The subtrees of \(t\) have a height less than or equal to \(h-1\). By induction hypothesis, each of the subtrees has a size of at most \(2^{h-1}\). Since there are 2 such subtrees, each with a maximum size of \(2^{h-1}\), the size of \(t\) is at most \(2 \times 2^{h-1} = 2^h\).

Q.E.D.
Problem 3
Consider the following BNF specification:

\[
\text{<LambdaExp> ::= } \text{<Identifier> }
\]

| \( \lambda ( \text{<Identifier> } \text{<LambdaExp> } ) \) |
| \( ( \text{<LambdaExp> <LambdaExp> } ) \) |

Use structural induction on \text{<LambdaExp>} to prove that if \text{e} \in \text{<LambdaExp>}, then \text{e} has the same number of left and right parentheses.

Solution:

Base case \text{le \equiv id}: #( == #) = 0

Inductive step 1, \text{le \equiv (\lambda ( id ) le_1)}:

Assume the result holds for \( k < n \). Let \( k_1 = #( \kappa_2 = #) \) of \text{le_1} with \( k_1 < n \), \( k_2 < n \). By assumption hypothesis, we have \( k_1 \equiv k_2 \). Therefore, \( n_1 = k_1 + 2 \equiv k_2 + 2 = n_2 \).

Inductive step 2, \text{le \equiv (le_1 le_2)}:

Assume the result holds for \( k < n \). Let \( k_{11} = #( \text{and } k_{12} = #) \) of \text{le_1}, \( k_{21} = #( \text{and } k_{22} = #) \) of \text{le_2} with \( k_{11} < n \), \( k_{12} < n \), \( k_{21} < n \), \( k_{22} < n \). By assumption hypothesis, we have \( k_{11} \equiv k_{12} \) and \( k_{21} \equiv k_{22} \). Therefore, \( n_1 = k_{11} + k_{21} + 1 \equiv k_{12} + k_{22} + 1 = n_2 \).

Q.E.D.